Structural Optimization with Frequency Constraints

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This paper presents a design optimization algorithm for structural weight minimization with multiple frequency constraints. An optimality criterion method based on uniform Lagrangian density for resizing and a scaling procedure to locate the constraint boundary were used in optimization. Multiple frequency constraints of equality and inequality types were addressed. The effectiveness of the algorithm was demonstrated by designing a number of truss structures with as many as 489 design variables. No attempt was made to reduce the number of design variables by such procedures as linking and/or invoking symmetry conditions. The design examples include a 10-bar truss, 200-bar truss, a modified ACOSS-II, and COFS (Control of Flexible Structures) mast truss. All the structures contain nonstructural mass besides their own mass. The algorithm is extremely stable and, in all cases, the optimum designs were obtained in less than 20 iterations regardless of the size of the structure and the number of design variables.

Introduction

THE optimal design of structures with frequency constraints is extremely useful in manipulating the dynamic characteristics in a variety of ways. For example, in most low-frequency vibration problems, the response of the structure to dynamic excitation is primarily a function of its fundamental frequency and mode shape. In such cases, the ability to manipulate the selected frequency can significantly improve the performance of the structure. Similarly, the aeroelastic characteristics of an aircraft wing are governed primarily by its torsional and bending properties, which can best be studied by the lower torsional and bending modes. In fact, in most narrow-band excitation problems, control of the frequencies in the critical range is tantamount to control of the dynamic response.

A large amount of literature related to the optimization of structures with frequency constraints is available. Most of these papers present design algorithms based on a single frequency constraint, which uses only the information about the fundamental frequency and the associated vibration mode.[1-6] A few papers however, consider multiple frequency constraints.[7-11] A brief description of the previous research work is given below. This is by no means the complete literature on frequency optimization.

Miura and Schmit[1] minimized the structural mass of a cantilever beam and a wing structure with a fundamental frequency constraint. A mathematical programming method based on an interior penalty function was used. This method can be used in conjunction with design variable linking for optimizing practical structures. Reference 2 considered the maximization of the fundamental frequency of a vibrating beam with a specified volume constraint. References 3–9 employed optimality criteria methods in designing minimum weight structures. Minimization of structural mass subject to a fundamental frequency constraint and maximization of the fundamental frequency for a given structural mass have been considered in the design of rods and trusses by Venkayya and Tischler.[6] Elwany and Barr[7] presented results for the minimum mass design of a cantilever beam subjected to multiple (up to three) torsional frequency constraints. A variational procedure was used in deriving the necessary conditions of optimal design. The resulting integral equations and their solution are continuous functions of spatial variables. Fleury and Sander[8] used a generalized optimality criterion method for designing beams with multiple frequency and buckling constraints. Basically, the method is an extension of the conventional optimality criterion approach, or it can be viewed as a linearization in mathematical programming. Khot[9] has considered a truss structure with two frequency constraints and found interesting anomalies in the convergence. Pedersen[10] has addressed beam problems with constraints on the first four frequencies using a linear programming method. Olhoff[11] presented results for maximizing a higher-order frequency subject to a given volume constraint. He uses a variational approach and derives the optimality condition in an integral form. The resulting optimum shape is a continuous function of the spatial variable. Reference 12 considered optimum structural design with fundamental frequency, stress, and displacement constraints and also compared the relative efficiency of different optimization techniques.

The present paper examines the problem of finding the minimum weight structure satisfying the frequency constraints. The design problems considered are optimization 1) with a single (fundamental or higher) frequency constraint (an equality constraint) and 2) with constraints on multiple frequencies (treated as equality and inequality constraints). The optimality criterion method in conjunction with a scaling procedure has been used to design a minimum weight structure with frequency constraints. The algorithm developed by Venkayya and Tischler[6] has been used as the basis for the development of the present multiple frequency constraint formulation.

This algorithm, based on an optimality criterion approach, is very efficient and reliable, even in the presence of a large number of design variables. The optimality criterion is derived by differentiating the Lagrangian with respect to the design variables, and the minimum weight design should satisfy this criterion. The essence of the optimality criterion is that at the

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optimum the weighted sum of the Lagrangian energy density must be the same in all the elements. The optimality condition consists of the gradients (constraints and objective function) and Lagrangian multipliers. The gradients of the frequency constraints and objective function can be derived analytically from the dynamic equilibrium equations and the expression for the objective function. In the case of a single constraint, the evaluation of the Lagrangian multiplier is simple and straightforward. However, in the case of multiple (active) constraints, determination of the Lagrangian multipliers requires approximations. The basic equations are nonlinear, and also the inadmissibility of nonpositive Lagrangian multipliers presents additional difficulties. In this work, Lagrangian multipliers are approximated using simple expressions derived from single constraint conditions. The optimization consists of three basic steps. The step is analysis of the structure followed by scaling and resizing. Analysis gives information about feasibility or nonfeasibility of the design. Scaling is for obtaining the feasible design. A simple scaling procedure is developed for multiple frequency constraints. An estimate of the scale factor is obtained to reach the frequency constraint boundary. The resizing algorithm is derived from optimality conditions.

This paper discusses the optimality algorithm, resizing procedure, and scaling techniques for multiple frequency constraints. The effectiveness of the algorithm is demonstrated by using large structural design applications with hundreds of design variables and multiple frequency constraints.

Problem Statement

The structural optimization problem is stated as follows. Minimize structural weight:

$$W(x) = \sum_{i=1}^{n} \rho_i x_i l_i \tag{1}$$

subject to behavior constraints

$$\begin{aligned} g_j(x) &= \omega_j^2 - \bar{\omega}_j^2 = 0 \qquad j = 1,2,\ldots,k \\ g_j(x) &= \omega_j^2 - \bar{\omega}_j^2 \geq 0 \qquad j = k+1, k+2,\ldots,m \end{aligned} \tag{2}$$

and side constraints on the design variables

$$x_i \geq x_i^\ell \tag{3}$$

where ρ_i is the density, l_i the length, x_i the design variable, x_i^ℓ the lower limit on the design variable, ω_j the jth natural frequency, $\bar{\omega}_j$ the specified value of the jth frequency, n the number of design variables, and m the number of design constraints.

Optimality Method

The optimality method consists of two steps. The first step involves derivation of the optimality conditions and associating them with an energy condition in the structure. In the second step, an iterative algorithm is derived with the help of the energy condition to achieve the optimality condition.

The Lagrangian formulation for the stated structural optimization problem is written as

$$L(\boldsymbol{x},\lambda) = W(x) - \sum_{j=1}^{m} \lambda_j g_j(\boldsymbol{x}) \tag{4}$$

where $L(x,\lambda)$ is the Lagrangian function and the λ are the Lagrangian multipliers. The unknowns are the design variables x_i and the Lagrangian multipliers λ_j.

Minimization of the Lagrangian L with respect to the design variable vector $\boldsymbol{x}$ gives the condition for the stationary value of the objective function as

$$\frac{\partial L}{\partial x_i} = \frac{\partial W(\boldsymbol{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \tag{5}$$

From Eq. (5), the optimality condition can be written as

$$\sum_{j=1}^{m} e_{ij}\lambda_j = 1 \qquad i = 1,2,\ldots,n \tag{6}$$

where

$$e_{ij} = \frac{\partial g_j / \partial x_i}{\partial W / \partial x_i} \tag{7}$$

Equation (7) represents the ratio of the constraint to the objective function gradients with respect to the design variables. These ratios can be associated with special forms of energy densities, depending on the type of constraints. For the case of frequency constraints, e_{ij} is computed as follows.

The eigenvalue problem corresponding to the undamped linear system is written as

$$K\phi_j = \omega_j^2 M \phi_j \tag{8}$$

where ω_j is the jth natural frequency, ϕ_j is the jth eigenvector or natural mode of the structure, and K and M are the stiffness and mass matrices, respectively. The mass matrix M is a combination of the structural mass M_s, and the nonstructural mass M_c.

In this paper, the natural frequencies are calculated using the Sturm sequence method[13,14] in conjunction with a bisection procedure. With this method, any desired frequencies and the corresponding modes can be computed without calculating any of the remaining frequencies and modes.

The square of the jth natural frequency of the structure can be defined by a Rayleigh quotient as

$$\omega_j^2(x) = \frac{\phi_j^t K \phi_j}{\phi_j^t M \phi_j} \tag{9}$$

where superscript t denotes the transpose. Substitution of Eq. (9) in Eq. (7) gives the expression for e_{ij} as

$$e_{ij} = \frac{\phi_j^t k_i \phi_j - \omega_j^2 \phi_j^t m_i \phi_j}{\rho_i x_i l_i} \tag{10}$$

where k_i and m_i are the ith element stiffness and mass matrices, respectively, in the global coordinate system. The eigenvectors were normalized using the expression

$$\phi_j^t M \phi_j = 1 \tag{11}$$

In deriving the e_{ij} expression, the stiffness and structural mass matrices were assumed to be linear functions of the design variables. This is true for truss and membrane elements, and needs minor modification for bending and other types of finite elements.

The solution of the optimization problem involves $(n+m)$ unknown quantities, where n is the number of design variables and m is the number of Lagrangian multipliers corresponding to m constraints. The optimality condition described in Eq. (6) gives n equations. The additional m unknowns are computed from the original constraint conditions.

Equation (10) can be rewritten as

$$e_{ij}\rho_i x_i l_i = \phi_j^t k_i \phi_j - \omega_j^2 \phi_j^t m_i \phi_j \tag{12}$$

Taking the summation of Eq. (12) over all the elements,

$$\sum_{i=1}^{n} e_{ij}\rho_i x_i l_i = \gamma_j \omega_j^2 \qquad j = 1,2,\ldots,m \tag{13}$$

where γ_j is the ratio of the nonstructural modal mass to the total modal mass and is given by

$$\gamma_j = \frac{\phi_j^t M_c \phi_j}{\phi_j^t M \phi_j} \tag{14}$$

Equation (13) for active constraints takes the form

$$\sum_{i=1}^{n} e_{ij}\rho_i x_i l_i = \gamma_j \bar{\omega}_j^2, \qquad j=1,2,\ldots \tag{15}$$

Combining Eqs. (6) and (15) provides the necessary equations for determining the Lagrangian multipliers as follows:

$$\sum_{i=1}^{n} e_{ij}\rho_i x_i l_i e_{ij} \lambda_j = \gamma_j \bar{\omega}_j^2, \qquad j=1,2,\ldots,m \tag{16}$$

or

$$H\lambda = \hat{G} \tag{17}$$

where the matrix H is given by

$$H = e^t D e \tag{18}$$

H is of the order $m\times m$ since it involves multiplication of $m\times n$, $n\times n$, and $n\times m$ matrices, and $\hat{G}$ is a vector of $\gamma_j\bar{\omega}_j^2$. D is a diagonal matrix, and its ith diagonal element is given by

$$D_{ii} = \rho_i x_i l_i \tag{19}$$

The elements of matrix H cannot be determined explicitly because the e and D matrices are functions of the design variable vector, which is itself unknown. Equations (6) and (17) are nonlinear sets of equations, and they have to be solved by iterative methods.

Optimization Technique

Resizing Algorithm

The optimality condition for multiple frequency constraints can be stated as: The weighted sum of the Lagrangian energy densities corresponding to multiple frequency constraints should be equal to unity in all of the elements. This can be achieved by satisfying the condition given in Eq. (6). By multiplying both sides of Eq. (6) by x_i^p and taking the pth root ($\alpha = 1/p$), we obtain

$$x_i^{k+1} = x_i^k \left[\sum_{j=1}^{m} C_j e_{ij}^k \right]^{\alpha} \tag{20}$$

where α controls the step size and C_j contains the weighting parameters that can be approximated as functions of the Lagrangian multipliers. Superscripts k and $k+1$ are introduced to denote the iteration numbers. The iteration scheme is repeated until the set convergence criterion is satisfied.

The design is started with an initial value of α as 0.5. After a design cycle [i.e., scaling the design to a feasible region and resizing the design using Eq. (20)] if the objective function increases from the previous iteration, then the α value is reduced by half. The smaller α value avoids large changes in the design when it is closer to the optimum and gives a stable convergence. It acts like move limits on the design variables. The lowest value for α is taken as 0.001. If α becomes smaller than 0.001, then the resizing is terminated.

Scaling Procedure

After each resizing step, the design variables are scaled to the feasible region to satisfy all of the design constraints. This helps in monitoring the progress of each design cycle and in finding the active constraints. The design variables are multiplied by a scaling factor Λ, and it is computed as follows:

The Rayleigh quotient for the jth frequency at the current design is written as

$$\omega_{jo}^2 = \frac{\phi_j^t K \phi_j}{\phi_j^t (M_s + M_c)\phi_j} \tag{21}$$

where ω_{jo} is the unscaled frequency. By scaling the design variables by a factor Λ_j, the desired frequency ω_{jd} is obtained. Then the Rayleigh quotient becomes

$$\omega_{jd}^2 = \frac{\Lambda_j \phi_j^t K \phi_j}{\phi_j^t (\Lambda_j M_s + M_c)\phi_j} \tag{22}$$

In this equation, it is assumed that scaling the structural members by Λ_j implies that K and M_s are scaled in a similar way. This is true in membrane elements and needs minor modification for bending-type elements. The nonstructural mass M_c does not depend on the design variables, and hence it is not multiplied by Λ_j. Equation (22) can be further simplified to relate the desired frequency ω_{jd} with the unscaled frequency ω_{jo} as

$$\omega_{jd}^2 = \frac{\Lambda_j}{\Lambda_j \eta_j + \gamma_j}\omega_{jo}^2 \tag{23}$$

where η_j is the ratio of the structural modal mass to the total modal mass and is given by

$$\eta_j = \frac{\phi_j^t M_s \phi_j}{\phi_j^t M \phi_j} \tag{24}$$

also,

$$\eta_j + \gamma_j = 1 \tag{25}$$

Now the ratio r_j^2 is defined as

$$r_j^2 = \frac{\omega_{jd}^2}{\omega_{jo}^2} = \frac{\Lambda_j}{\Lambda_j \eta_j + \gamma_j} \tag{26}$$

Then the scaling parameter corresponding to the jth frequency becomes

$$\Lambda_j = \frac{\gamma_j r_j^2}{1 - r_j^2 \eta_j} \quad \text{for } r_j^2 \eta_j < 1 \tag{27}$$

and

$$\Lambda_j = r_j^2 \quad \text{for } r_j^2 \eta_j \geq 1 \tag{28}$$

One scaling parameter Λ_j is computed for each constraint, and from these values one value is selected based on the type of constraints. If all are inequality constraints, then Λ is selected such that the most violated constraint becomes active. If any one of the constraints is an equality type, then the scaling parameter corresponding to that one is used. This formulation handles only one equality constraint, and the rest have to be inequality constraints. This is so because, after scaling, more than one constraint can never be made active unless it is a repeated frequency. Besides, if there are more than one equality type constraint, there may not exist a feasible solution. Since the frequency constraints are nonlinear functions of the design variables, repetition of the scaling procedure may be necessary in some design cycles.

Role of the Lagrangian Multipliers/Active and Passive Constraints

The Lagrangian multipliers are computed by using simple approximations. Here the λ are considered merely as weighting parameters, and they are computed as if there is only one constraint. This approximation eliminates solving the coupled nonlinear equations given in Eq. (17) and also the difficulties associated with the negative λ values.

The optimality condition for a single constraint is written as

$$e_i\lambda = 1 \qquad i = 1,2,\ldots,n \tag{29}$$

Substituting $e_i = 1/\lambda$ in Eq. (15) gives

$$\frac{1}{\lambda}\sum_{i=1}^{n}\rho_i x_i l_i = \gamma_j \bar{\omega}_j^2 \tag{30}$$

where λ takes the form

$$\lambda = \frac{W}{\gamma_j \bar{\omega}_j^2} \tag{31}$$

and W represents the weight of the structure. The jth Lagrangian multiplier corresponding to the jth frequency constraint can be approximated as

$$\lambda_j = \frac{W}{\bar{\omega}_j^2(1-\eta_j)} \tag{32}$$

It is evident from Eq. (32) that, in the absence of nonstructural mass, the value of the Lagrangian multiplier becomes indefinite. The structure cannot be optimized for frequency constraints alone unless some nonstructural mass is associated with the system. As a matter of fact, nonstructural mass is the major part of the total mass in most of the aerospace structures. Optimization in the absence of nonstructural mass is possible only when additional constraints such as static displacements or stresses are present.

The concept of active and passive constraints is crucial in the multiple frequency constraint problem. The active constraints are those within a certain percentage from their respective limits. The remaining constraints are passive. The resizing formula [Eq. (20)] uses the Lagrangian multipliers corresponding to the active constraints. The active and passive constraints decision was made at the outset (after the first iteration) of the design, and no changes were made during the rest of the design. The iteration was continued as long as there was improvement in the design (reduction in weight, for example). It is worthwhile pointing out some characteristics of the emerging designs during the iterations. If only a single Lagrangian multipler (corresponding to one active constraint) is participating in the iteration, the weight of the structure reduces rapidly and the step size adjustments (α value) have to be made very early in the design. This happens because the inactive frequencies tend to cluster toward the active frequency and mode switching can destroy smoother convergence. However, by adjusting the value of α, the optimum design can be approached more smoothly. The price of reducing α is reflected in an increased number of iterations. In the other extreme, when the number of active constraints are more because of the larger percentage specifications around the limiting values, then more Lagrangian multipliers participate in the resizing. Iterations using more Lagrangian multipliers produce two interesting characteristics: first, the problem tends to be overconstrained, and the resulting optimum corresponds to an upperbound solution. The second characteristic is that the clustering of frequencies (at least among the active ones) is significantly reduced. Also, the inequality constraints stay away from the constraint boundary. The step size (α) adjustments do not play a significant role because the iteration converges quite smoothly. The percentage value for finding the active constraints varies from problem to problem, but usually within 5–10% from the constraint limit would be adequate.

Optimization Algorithm

An outline of the design optimization procedure is given as follows:

1) The design starts with uniform sizes for all the design variables.

2) At the current design, the desired natural frequencies and the corresponding mode shapes are calculated using the Sturm sequence method.

3) The design variables are scaled to the constraint boundary in order to satisfy all the behavior constraints. The Lagrangian multipliers are evaluated.

4) The Lagrangian energy is computed for each mode in each element. $\Sigma e_{ij}\lambda_j$ is evaluated for all the elements. The design variables are resized using Eq. (20).

Repeat steps 2–4 until the set convergence requirements are satisfied. Steps 2–4 constitute one iteration. In Eq. (10), e_{ij} can become negative for some elements when the square of the natural frequency multiplied by the modal kinetic energy is larger than the modal potential energy. In this case, the corresponding design variable takes its lower limit value, and the e_{ij} contribution in Eq. (20) is zero. This creates aberrations and takes a few extra iterations in converging to the optimal solution. So, it is preferable to use only the modal potential energy for e_{ij} in all the elements. This is justified whenever the nonstructural mass is much greater than the structural mass. This provides a stable convergence in design optimization. All of the problems solved in this paper employed only the modal potential energy because of substantial nonstructural mass.

Numerical Results

The effectiveness of the optimality algorithm is demonstrated by using a 10-bar truss (Fig. 1), a 200-bar truss (Fig. 4), a modified ACOSS-II (Fig. 6), and a COFS model (Fig. 9) as

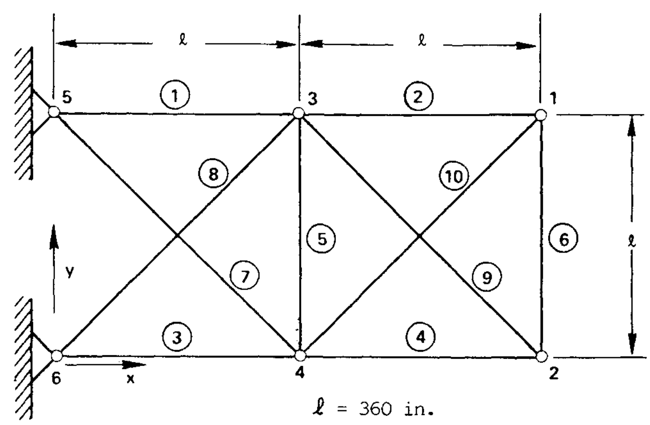

Fig. 1 Ten-bar truss structure.

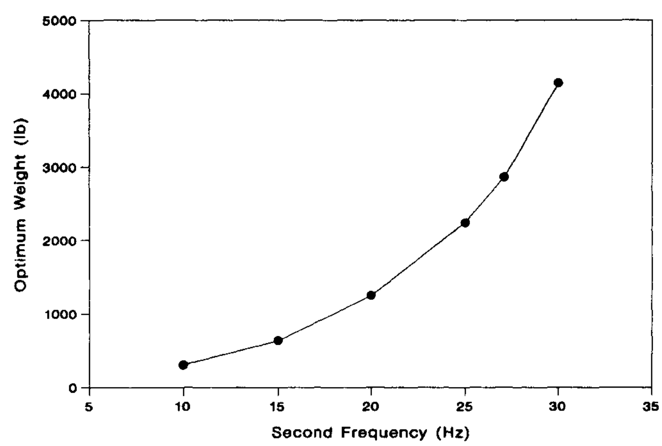

Fig. 2 Ten-bar truss—specified second frequency.

design examples. The emphasis here is in solving large-scale structural design problems with hundreds of design variables. In the results presented, one iteration of optimization represents one eigenvalue analysis.

Ten-Bar Truss

The structure shown in Fig. 1 is made up of aluminum with Young's modulus, 10^7 psi and weight density, 0.1 lb/in.^{3}. At each of the four free nodes, 2.588 lb-s^2/in. nonstructural mass is added. One design variable was assigned to each structural element. At the initial design, all the cross-sectional areas were 1.0 in.^{2} for each member. The design was scaled by a factor of 9.5318 to obtain a 4000-lb initial weight. The lower limit on the design variables was 0.1 in.^{2}.

Reference 6 presents optimization studies with different fundamental frequency limits. In this paper, first the struture

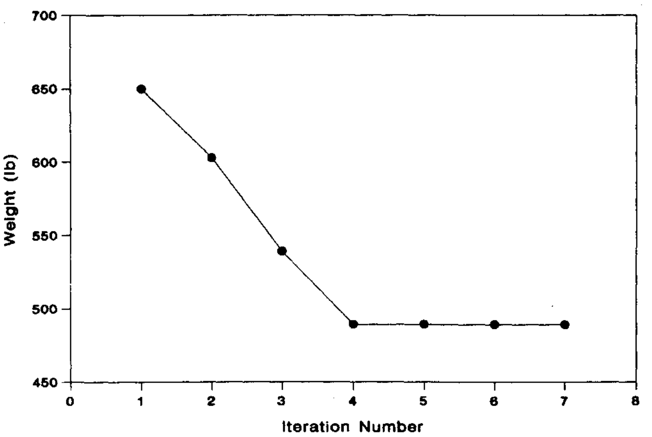

Fig. 3 Ten-bar truss—three inequality constraints.

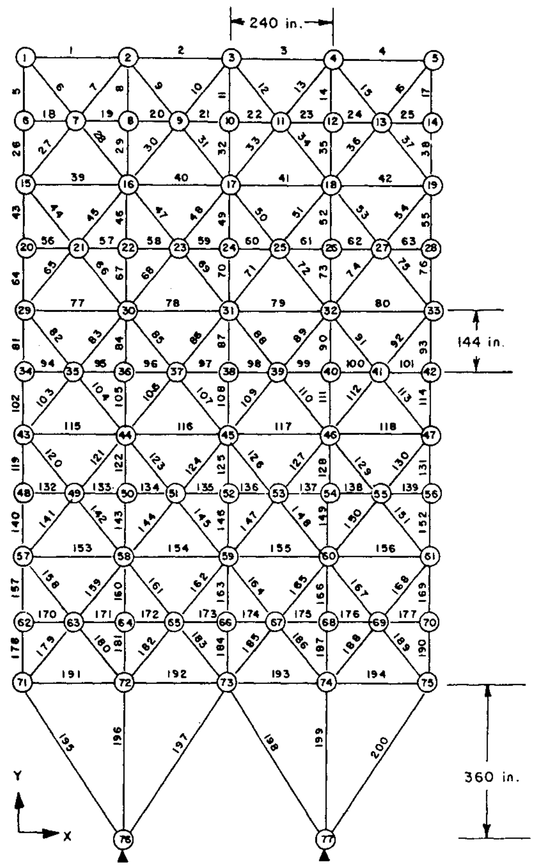

Fig. 4 200-bar truss structure.

Table 1 Ten-bar truss: initial and final frequencies for a specified second frequency, Hz

Frequency no.	Initial design	$\omega_2 = 10.0$	$\omega_2 = 15.0$	$\omega_2 = 20.0$	$\omega_2 = 25.0$	$\omega_2 = 27.08$	$\omega_2 = 30.0$
1	08.96	03.26	04.92	06.64	08.40	09.17	10.34
2	27.08	10.00	15.00	20.00	25.00	27.08	30.00
3	27.45	10.19	15.07	20.13	25.00	27.11	30.13
4	51.25	16.01	15.30	21.51	27.66	30.96	35.42
5	58.00	18.08	22.21	30.39	39.34	43.96	50.73
6	64.73	22.96	24.28	32.81	41.32	45.54	51.39
7	66.87	25.21	39.49	52.53	65.35	70.40	78.90
8	80.85	27.25	41.64	55.89	70.49	76.84	87.71
Weight, lb	4000.0	304.5	637.0	1251.5	2243.8	2865.9	4143.9

Table 2 Ten-bar truss: optimum design variables, in.^{2}, for a specified second frequency, Hz

Frequency limit	10.0	15.0	20.0	25.0	27.08	30.0
Element no.						
1	0.910	2.313	4.435	7.699	9.598	13.720
2	0.821	2.154	4.140	7.224	8.979	12.866
3	0.910	2.313	4.435	7.697	9.598	13.720
4	0.821	2.154	4.140	7.223	8.979	12.866
5	0.768	0.602	1.223	2.195	2.905	3.907
6	0.570	0.353	0.760	1.382	1.850	2.774
7	0.712	1.723	3.413	6.211	7.898	11.075
8	0.712	1.723	3.413	6.211	7.898	11.075
9	0.581	1.037	2.114	4.017	5.431	8.460
10	0.581	1.036	2.114	4.003	5.431	8.460
Weight, lb	304.5	637.0	1251.5	2243.8	2865.9	4143.9

Table 3 Ten-bar truss: initial and final frequencies, Hz, in different constraint conditions

Frequency no.	Initial design	$\omega_1 = 7.0$	$\omega_1 = 10.0$	$\omega_1 = 7.0$ $\omega_2 \geq 15.0$	$\omega_1 = 10.0$ $\omega_2 \geq 15.0$	$\omega_1 = 7.0$ $\omega_2 \geq 15.0$ $\omega_3 \geq 20.0$	$\omega_1 \geq 3.5$ $\omega_2 \geq 10.0$ $\omega_3 \geq 14.0$
1	8.96	7.00	10.00	7.00	10.00	7.00	4.40
2	27.08	10.96	13.73	15.58	19.16	15.61	12.14
3	27.45	16.27	22.29	16.93	24.52	20.17	14.00
4	51.25	18.21	25.19	18.75	27.16	20.77	17.89
5	58.00	27.39	38.04	29.13	38.71	28.76	19.58
6	64.73	29.55	42.21	30.30	40.53	29.76	22.96
7	66.87	47.92	65.79	46.93	67.66	53.88	34.01
8	80.85	50.34	69.93	49.67	71.38	56.03	35.72
Weight, lb	4000.0	1137.3	2614.0	1172.6	2736.3	1308.4	489.17

was designed with a higher-order frequency constraint alone. The structural weight was optimized with the second natural frequency constrained to six different values. The second frequency was required to be 10, 15, 20, 25, 27.08, and 30 Hz in six different cases. Ten design variables and one frequency constraint were considered. Table 1 presents all the frequencies and the structural weight at the optimum design in all these cases. Table 2 gives the optimum design variables. The variation of the optimum weight with the second frequency limits is shown in Fig. 2. From the results obtained, it is clear that the optimality algorithm can be effectively used in designing structures with a higher-order frequency constraint.

After this, the 10-bar truss was designed with multiple frequency constraints. The structural weight was minimized such that the fundamental, second, and third frequencies were constrained. Tables 3 and 4 present the results obtained under different constraint conditions. The first set was with $\omega_1 = 7.0$ Hz, the second with $\omega_1 = 10.0$ Hz, the third with $\omega_1 = 7.0$, $\omega_2 \geq 15.0$ Hz, the fourth with $\omega_1 = 10.0$, $\omega_2 \geq 15.0$ Hz, the fifth with $\omega_1 = 7.0$, $\omega_2 \geq 15.0$, and $\omega_3 \geq 20.0$ Hz and, finally, all inequality constraints, $\omega_1 \geq 3.5$, $\omega_2 \geq 10.0$, and $\omega_3 \geq 14.0$ Hz were considered. Table 3 gives all the frequencies at the initial and optimum designs, and Table 4 consists of all the design variables at the optimum. The iteration history for $\omega_1 \geq 3.5$, $\omega_2 \geq 10.0$, and $\omega_3 \geq 14.0$ Hz is shown in Fig. 3. It took seven iterations to reach the optimal solution. In the last three iterations, there was not much change in the objective function.

200-Bar Truss

The 200-bar truss was formulated by Venkayya et al.[15] for demonstrating the computational efficiency of the optimality method in designing very large structures. The structure shown in Fig. 4 has 77 nodes, 200 bar elements, and 150 deg of freedom. At each of the 75 free nodes, 0.5 lb-s^2/in. nonstructural mass was considered. The structure is made up of steel with Young's modulus, 30.0×10^6 psi and a weight density of 0.283 lb/in.^{3}. Two hundred design variables were assigned to the 200 bar elements in structural weight optimization. At the initial design, all the cross-sectional areas were 10.0 in.^{2}. The lower limit on the design variables was 0.1 in.^{2}. The initial weight of the structure was 99634 lb, and the first five frequencies at the initial designed were 2.87, 13.02, 13.72, 31.92, and 37.68 Hz. First, the structure was designed with a fundamental frequency constraint. Various fundamental frequency limits were considered starting from 2.0 to 8.0 Hz. Table 5 presents the first five frequencies and the optimum weight for different fundamental frequency limits. Figure 5 shows the variation of the optimum weight as the frequency limit is increased. The structural weight increases very rapidly at the higher frequency limits. Beyond a certain frequency limit, it is not possible to design a practical structure. Next, this truss was designed with two frequency constraints. $\omega_1 = 3.0$ Hz and $\omega_2 \geq 9.0$ Hz. The optimum weight of 5226.5 lb, was obtained in seven iterations.

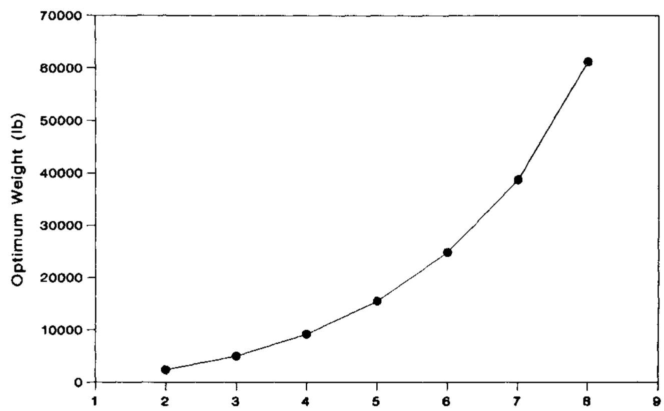

Fig. 5 200-bar truss—first frequency constraint.

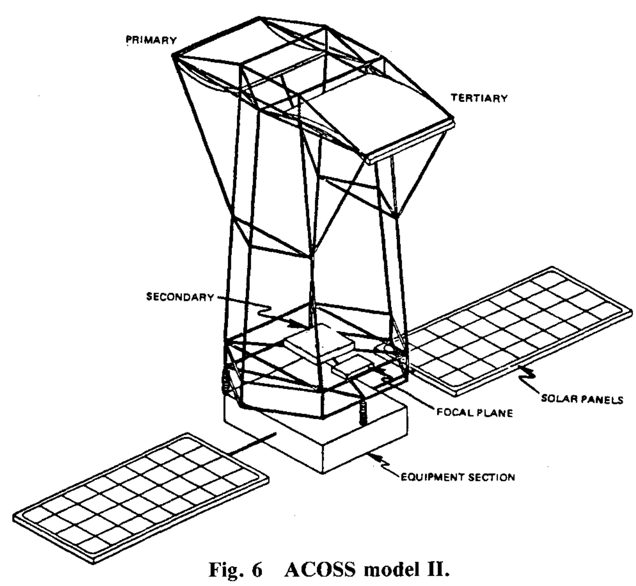

Fig. 6 ACOSS model II.

The first five frequencies at the optimum were 3.0, 9.01, 9.41, 14.39, and 14.78 Hz. Also, the same structure was designed with $\omega_1 = 4.0$ and $\omega_2 \geq 10.0$ Hz. The optimum weight was 9220.9 lb, and the first five frequencies at the final design were 4.0, 10.0, 14.46, and 15.78 Hz. In both the cases, the second frequency constraint was satisfied as if it were an equality constraint. The design variables at the optimum for both the cases are given in Ref. 16.

Table 4 Ten-bar truss: optimum design variables, in.^{2}, in different constraint conditions

Element no.	$\omega_1 = 7.0$	$\omega_1 = 10.0$	$\omega_1 = 7.0$ $\omega_2 \geq 15.0$	$\omega_1 = 10.0$ $\omega_2 \geq 15.0$	$\omega_1 = 7.0$ $\omega_2 \geq 15.0$ $\omega_3 \geq 20.0$	$\omega_1 \geq 3.5$ $\omega_2 \geq 10.0$ $\omega_3 \geq 14.0$
1	6.045	13.965	5.511	13.147	5.672	2.306
2	1.969	4.437	1.937	5.683	3.823	1.304
3	6.045	13.965	5.511	13.147	5.672	2.306
4	1.969	4.437	1.937	5.683	3.823	1.304
5	0.100	0.100	0.207	0.488	0.646	0.639
6	0.100	0.100	0.414	0.517	0.321	0.557
7	3.206	7.579	3.616	9.093	4.191	1.029
8	3.206	7.579	3.616	9.093	4.191	1.029
9	2.226	5.009	2.414	4.110	1.604	0.800
10	2.226	5.009	2.414	4.110	1.604	0.800
Weight, lb	1137.3	2614.0	1172.6	2736.3	1308.4	489.7

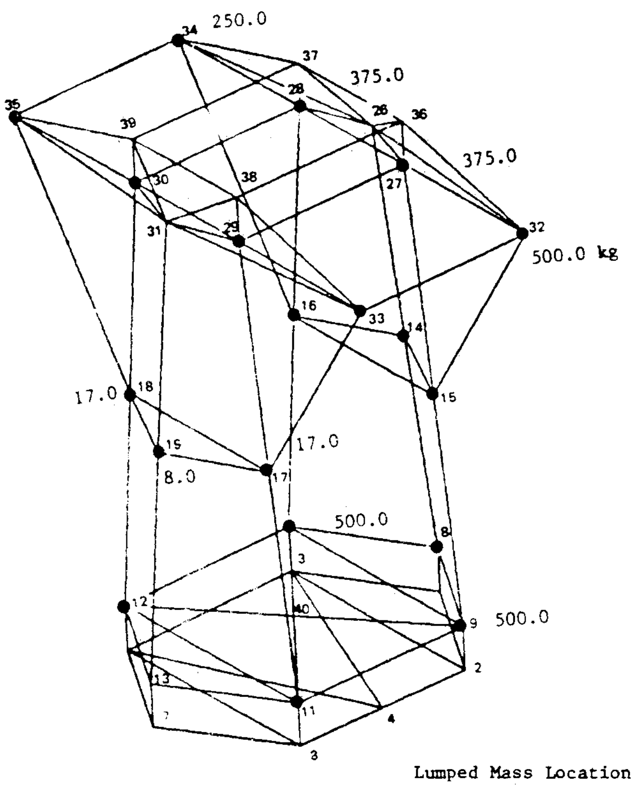

Fig. 7 Finite-element representation of ACOSS II.

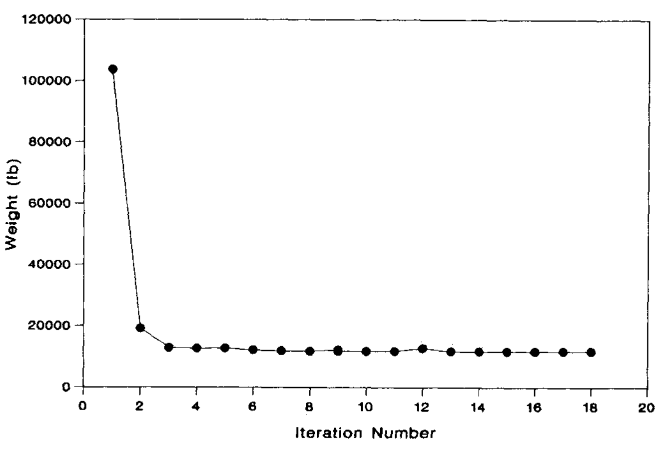

Fig. 8 ACOSS II iteration history with two constraints.

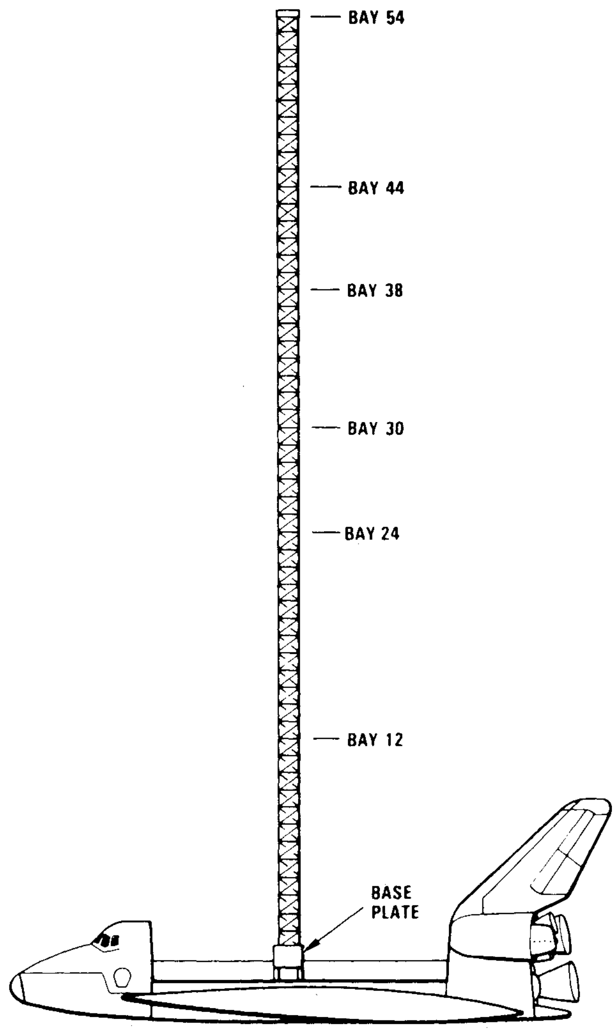

Fig. 9 COFS model.

Modified ACOSS-II

The Active Control of Space Structures (ACOSS) model II was developed by the C.S. Draper Laboratory.[17] The structure consists of two subsystems: the optical support structure and the equipment section, which are connected by springs at three points to allow vibration isolation (Fig. 6). In this paper, ACOSS-II was modified to have only the optical support structure, and it is fixed at the three springs locations. Also, a couple of nodes were removed from the original model. The

Table 5 200-bar truss: initial and final five frequencies, Hz, for a specified fundamental frequency

	ω_1	ω_2	ω_3	ω_4	ω_5	Weight, lb
Initial design	2.87	13.02	13.72	31.92	37.68	99634.0
Frequency limit						
2.0	2.00	6.24	6.66	10.69	12.64	2365.7
3.0	3.00	8.19	8.92	13.67	13.81	5004.7
4.0	4.00	9.79	10.90	14.43	15.66	9198.4
5.0	5.00	11.24	12.71	15.13	17.17	15500.9
6.0	6.00	12.61	14.28	15.90	18.13	24830.6
7.0	7.00	13.93	15.86	16.90	19.65	38683.3
8.0	8.00	16.74	19.08	20.15	24.16	61186.7

modified ACOSS-II consists of the upper mirror support truss, the lower mirror support truss, and the metering truss. The finite element model is shown in Fig. 7; it has 33 nodes and 113 truss elements. All of the elements are made up of graphite/epoxy with Young's modulus, 18.5×10^6 psi and weight density, 0.055 lb/in.^{3}. The lower limit on the design variables was 0.1 in.^{2}. The initial cross-sectional areas were taken as 10.0 in.^{2}. At the initial design, the weight was 18709.8 lb, and the first five frequencies were 1.21, 2.71, 4.21, 10.33, and 10.48 Hz.

The modified ACOSS-II was designed with 113 design variables under two different sets of frequency constraints. In the first set, the structural weight was minimized such that the fundamental frequency was equal to 2.0 Hz. The optimum weight was 11687.3 lb, and the first five frequencies were 2.00, 2.04, 2.39, 5.19, and 5.68 Hz. The second frequency is very close to the fundamental frequency. In the second set, the constraints were $\omega_1 = 2.0$ and $\omega_2 \geq 3.0$ Hz. The optimum weight was 11820.2 lb, and the first five frequencies were 2.00, 3.72, 4.46, 6.78, and 7.22 Hz. The structural weight was increased by 133 lb (approximately 1%) to separate the second frequency from the fundamental one. All the five frequencies were well separated. Figure 8 shows the iteration history for this case. A large reduction in the weight was obtained in the first four iterations, and afterward the reductions in weight were very small. Reference 16 presents the design variable values at the optimum. In this example, the second frequency tries to approach the first one very rapidly, so both the Lagrangian multipliers were used throughout the optimization.

COFS Model

The Mast Flight System[18] is composed of several subsystems. The primary structural component is the beam assembly (Fig. 9), which is called the Control of Flexible Structures (COFS) model. The beam cross section is triangular, with the longerons located at the vertices of an equilateral triangle. The truss structure repeats itself in two-bay segments. There are 27 two-bay segments, for a total of 54 bays. A typical two-bay segment of the truss structure is shown in Fig. 10.

The structure consists of 489 truss elements; thus, 489 design variables were considered in the design optimization. All of the elements are made up of graphite/epoxy with Young's modulus, 18.5×10^6 psi, and weight density 0.055 lb/in.^{3}. The initial cross-sectional areas were taken as 1.0 in.^{2}, and the lower limit on the design variables was 0.1 in.^{2}. The structural weight was minimized such that the fundamental frequency was equal to 0.18 Hz. At the initial design, the weight was 1407.42 lb, and the first five frequencies were 0.2058, 0.2058, 1.4570, 1.4570, and 3.6524 Hz. Figure 11 gives the mode shapes corresponding to the first, third, and fifth frequencies. The second and fourth modes are similar to the first and third, respectively, because of the symmetry in the structure. The optimum weight was 258.35 lb, and the first four frequencies were 0.1800, 0.1802, 1.1068, and 1.1221 Hz.

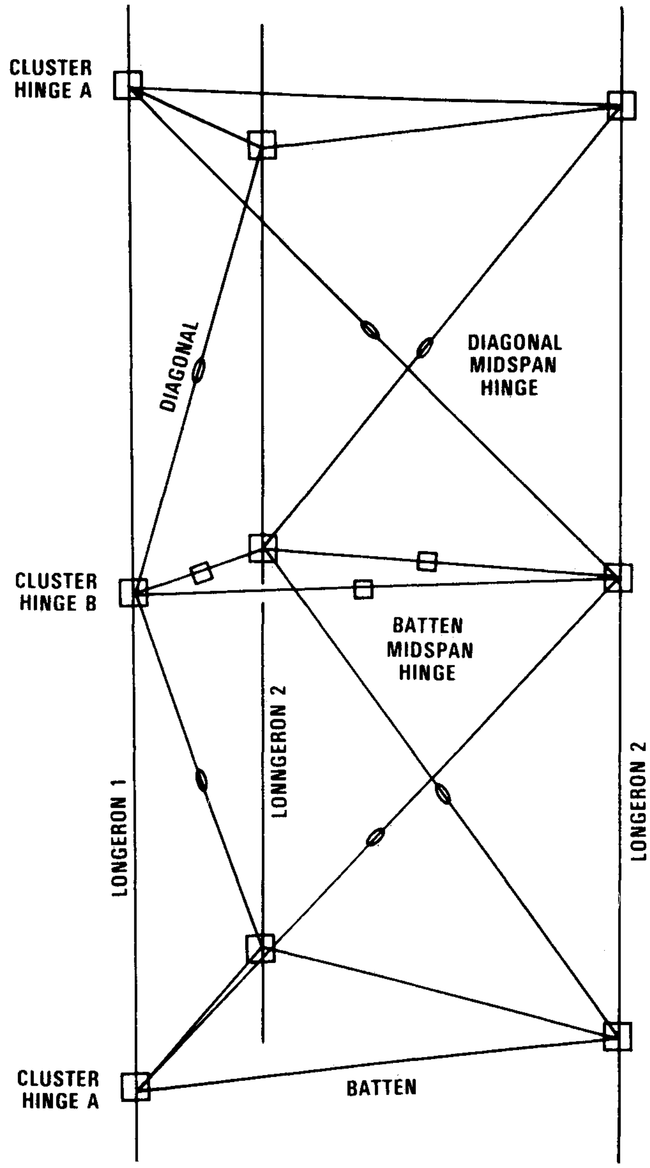

Fig. 10 Two-bay representation.

The second frequency is not equal to the first and, similarly, the fourth is not equal to the third. The optimization algorithm created unsymmetry in the structure; thus, the repeated frequency condition is eliminated. Most of the design variables reached their lower limit, and the longerons contributed most to the weight. The bottom longerons are much stronger than the top ones. The details of the optimum design are given in Ref. 16.

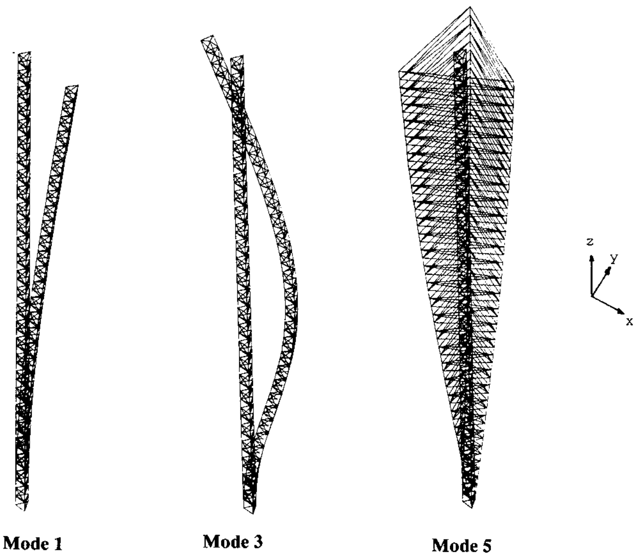

Fig. 11 COFS mode shapes.

Conclusions

In this paper, the minimum weight design of structures with multiple frequency constraints was obtained using an optimality method. A simple resizing scheme in conjunction with a scaling procedure has been used. Once again, the optimality criterion method has shown to be very efficient in designing large practical structures with a small number of iterations. The number of design variables considered was very large. All of the structures contain nonstructural mass besides their own mass. In this paper, results are presented for four truss structures, and Ref. 16 contains some additional truss and wing-type structures.

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